

Chiral Symmetry in Light-Cone Field Theory

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An analysis of spontaneously broken chiral symmetry in light-cone field theory is presented. The non-locality inherent to light-cone field theory requires revision of the standard procedure in the derivation of Ward-Takahashi identities. We derive the general structure of chiral Ward-Takahashi identities and construct them explicitly for various model field theories. Gell-Mann–Oakes–Renner relations and relations between fermion propagators and the structure functions of Nambu-Goldstone bosons are discussed and the necessary modifications of the Ward-Takahashi identities due to the axial anomaly are indicated.

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I. INTRODUCTION

Analysis of the symmetry properties constitutes the most powerful tool in the study of physical systems. Symmetry properties constrain the structure of the theoretical description and have far reaching consequences for the spectrum of states of the system. In systems with infinitely many degrees of freedom, symmetries can be realized in different ways. Very often, the different phases of physical systems are characterized by their realization of a symmetry. Of great importance is the case in which a continuous symmetry is not realized by the ground state of the system, *i.e.*, if the symmetry is spontaneously broken. The Goldstone theorem asserts that in this case the spectrum exhibits massless excitations. It thereby connects properties of the ground state of the system with the existence of massless Nambu-Goldstone (NG) particles [1, 2]. In the light-cone formulation of a field theory exhibiting the phenomenon of spontaneous symmetry breakdown the connection between ground state properties and the spectrum of excitations appears to be lost (some early studies within the parton model can be found in [3, 4, 5]; for a review of light-cone quantization, see [6]). The ground state is determined kinematically, its properties are independent of the dynamics, in particular they are independent of the realization of the symmetry. Despite this deficiency in characterizing the system by the structure of the ground state, light-cone formulations of model field theories like the chiral Gross-Neveu [7], the 't Hooft [8] or the Nambu–Jona-Lasinio [9] model exhibit massless particles. Similarly, if string theory is solved in light-cone quantization, the spectrum of excitations is correctly predicted. Also the successful application of light-cone field theory in the analysis of deep inelastic scattering on strongly interacting systems which are built upon the non-trivial QCD ground state indicates compatibility of the kinematical light-cone vacuum with non-trivial dynamics.

In this work we study symmetry properties of systems with spontaneously broken chiral symmetry. As in standard field theory, we will formulate the symmetry prop-

erties as chiral Ward-Takahashi (WT) identities [10, 11]. Such an analysis is most conveniently carried out with the help of functional techniques. In this framework we define a fermionic light-cone field theory by the generating functional containing only the unconstrained degrees of freedom, *i.e.*, with the “bad” components of the spinors eliminated. Such a fermionic theory is non-local, irrespective of the dynamics. This non-locality reflects the dependence of the light-cone energy p_+ on the inverse of the light-cone momentum p_- in the dispersion relation of a free particle

$$p_+ = \frac{m^2 + p_\perp^2}{2p_-}.$$

Kinematical nature of the vacuum and non-locality of light-cone field theory have the same origin.

In the standard derivation of the relevant WT identities one considers soft space-time dependent modulations of the global symmetry transformations [12]. In the symmetry broken phase, these transformations generate excitations with excitation energies approaching zero with increasing wavelength of the modulations. Via the WT identities one identifies the creation operator of the NG bosons in the limit of vanishing momentum and one establishes a series of relations between the composite NG bosons and the fundamental fermionic degrees of freedom. In application to light-cone theory, this standard procedure fails due to the non-locality of the light-cone Hamiltonian. The infinite range of the non-locality induced by the inverse of p_- destroys the connection between soft modulations and low excitation energies. This is akin to the suppression of NG particles in the presence of long-range interactions in gauge theories. We will demonstrate this failure of the standard procedure and show that the excitation of massive particles is not suppressed in the limit of long-wavelength modulations. In order to adjust the procedure to the light-cone Hamiltonian non-local transformations of the fermion fields have to be considered. We will carry out such an analysis by deriving the general form of the light-cone WT identities and by explicitly constructing the non-local symme-

try transformations for various model field theories. In addition to the derivation of Gell-Mann–Oakes–Renner (GOR) like relations [13] we consider the connection between fermion propagator and the structure function of the NG bosons [12].

II. FERMIONIC LIGHT-CONE FIELD THEORIES

In this section we will consider fermionic theories described by a Lagrangian of the following structure,

$$\mathcal{L} = \bar{\psi}(iD_\mu\gamma^\mu - m)\psi + \mathcal{L}_{\text{int}}(\bar{\psi}, \psi). \quad (1)$$

The covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu \quad (2)$$

couple the fermions to a gauge field which might be either external or dynamical. In the latter case integration over the gauge field has to be performed. The Lagrangian (1) may also include a fermionic self-interaction, \mathcal{L}_{int} . The expression for \mathcal{L} may contain implicitly sums over fermion species (“color”) while flavor dependences important in phenomenological applications are irrelevant for our discussion. We use a representation of the Dirac matrices particularly suited to the light-cone approach,

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \alpha_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \quad (3)$$

Then, γ_5 and the projection operators Λ^\pm are given by

$$\gamma_5 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \Lambda^\pm = \frac{1 \pm \gamma^0\gamma^3}{2}, \quad \gamma^0\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

The projection operators Λ^\pm decompose the 4-spinor into 2-spinors,

$$\psi = \frac{1}{2^{1/4}} \begin{pmatrix} \varphi \\ \chi \end{pmatrix},$$

and the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= i\varphi^\dagger D_+\varphi + i\chi^\dagger D_-\chi \\ &+ \frac{i}{\sqrt{2}} (\chi^\dagger D_m\varphi - \varphi^\dagger D_m^\dagger\chi) + \mathcal{L}_{\text{int}}(\varphi, \chi) \end{aligned} \quad (5)$$

with

$$D_m = \sigma_3 D_1 + iD_2 - i\sigma_1 m. \quad (6)$$

Only the spinor φ is dynamical since no time derivative of χ is present. In canonical quantization, χ is treated as a constrained field. This reduction in the number

of dynamical degrees of freedom makes the single particle states with given momentum unique and thereby the light-cone vacuum trivial. In the representation (4), chiral rotations are defined by

$$\varphi \rightarrow e^{i\alpha\sigma_3}\varphi, \quad \chi \rightarrow e^{i\alpha\sigma_3}\chi. \quad (7)$$

The light-cone components of the associated axial current are given by

$$\begin{aligned} j_5^+ &= \varphi^\dagger\sigma_3\varphi, & j_5^- &= \chi^\dagger\sigma_3\chi, \\ j_5^1 &= \frac{1}{\sqrt{2}}(\varphi^\dagger\chi + \chi^\dagger\varphi), & j_5^2 &= \frac{i}{\sqrt{2}}(\chi^\dagger\sigma_3\varphi - \varphi^\dagger\sigma_3\chi). \end{aligned} \quad (8)$$

We also note that in this representation, the relevant fermion bilinears read

$$\begin{aligned} \bar{\psi}i\gamma_5\psi &= -\frac{1}{\sqrt{2}}(\varphi^\dagger\sigma_2\chi + \chi^\dagger\sigma_2\varphi), \\ \bar{\psi}\psi &= -\frac{1}{\sqrt{2}}(\varphi^\dagger\sigma_1\chi + \chi^\dagger\sigma_1\varphi). \end{aligned} \quad (9)$$

With the choice of the four-fermion interaction

$$\begin{aligned} \mathcal{L}_{\text{int}} &= \frac{g^2}{2} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] \\ &= \frac{g^2}{4} [(\varphi^\dagger\sigma_1\chi + \chi^\dagger\sigma_1\varphi)^2 + (\varphi^\dagger\sigma_2\chi + \chi^\dagger\sigma_2\varphi)^2], \end{aligned} \quad (10)$$

the Lagrangian of Eq. (1) without the gauge field is that of the (one flavour) Nambu–Jona-Lasinio (NJL) model [9]. It is invariant under chiral rotations provided the (bare) mass m vanishes.

In the standard approach to light-cone quantization one employs the canonical formalism after eliminating the constrained variables [6]. For investigating issues related to symmetries the path integral provides a more appropriate framework. We start our investigations with the generating functional

$$\begin{aligned} Z[\eta, \gamma] &= \int D[\varphi, \chi] \\ &\exp \left\{ i \int d^4x (\mathcal{L} + \eta^\dagger\varphi + \varphi^\dagger\eta + \gamma^\dagger\chi + \chi^\dagger\gamma) \right\}. \end{aligned} \quad (11)$$

We define fermionic light-cone field theories as effective theories obtained by integrating out the “constrained” fields χ . This is the quantum mechanical version of eliminating the constrained variables by solution of the associated Euler-Lagrange equations. In integrating out χ all quantum fluctuations of these variables are kept. Both the kinetic terms and the four-fermion interaction of the NJL model are quadratic in χ . Therefore these variables can be eliminated by evaluating Gaussian integrals with the result

$$Z[\eta, \gamma] = \int D[\varphi] \exp \{ iS[\varphi] + iS[\eta, \gamma] \}. \quad (12)$$

Action and source terms are given by

$$S[\varphi] = \int d^4x (i\varphi^\dagger D_+\varphi + i\chi^\dagger D_-[\varphi]\chi) + i \text{tr} \ln D_-[\varphi], \quad (13)$$

$$s[\eta, \gamma] = \int d^4x \left(\eta^\dagger \varphi + \varphi^\dagger \eta + \gamma^\dagger \chi + \chi^\dagger \gamma + \gamma^\dagger \frac{i}{D_-[\varphi]} \gamma \right). \quad (14)$$

The action contains the composite field

$$\chi = -\frac{1}{\sqrt{2}D_-[\varphi]} D_m \varphi. \quad (15)$$

In the generating functional, we have retained the associated source γ . The field dependent covariant derivative is given by

$$D_-[\varphi] = D_- + ig^2 \begin{pmatrix} \varphi_2^\dagger \varphi_2 & \varphi_2^\dagger \varphi_1 \\ \varphi_1^\dagger \varphi_2^\dagger & -\varphi_1^\dagger \varphi_1 \end{pmatrix} \quad (16)$$

and acts on the source γ and the spinors

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2^\dagger \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2^\dagger \end{pmatrix}. \quad (17)$$

For most of our following studies it will be convenient to eliminate the four-fermion interaction in favor of two auxiliary bosonic fields, σ and π . To this end we rewrite the generating functional (11) as

$$Z[\eta, \gamma] = \int D[\varphi, \chi, \sigma, \pi] \exp \left\{ iS[\varphi, \chi, \sigma, \pi] + i \int d^4x (\eta^\dagger \varphi + \varphi^\dagger \eta + \gamma^\dagger \chi + \chi^\dagger \gamma) \right\},$$

with the action

$$S[\varphi, \chi, \sigma, \pi] = \int d^4x \left(i\varphi^\dagger D_+ \varphi + i\chi^\dagger D_- \chi + \frac{i}{\sqrt{2}} (\chi^\dagger D_\perp \varphi - \varphi^\dagger D_\perp^\dagger \chi) - \frac{1}{2}(\sigma^2 + \pi^2) \right)$$

and

$$D_\perp = \sigma_3 D_1 + iD_2 - ig(\sigma\sigma_1 + \pi\sigma_2) - im\sigma_1. \quad (18)$$

Integrating out the constrained variables χ the generating functional becomes

$$Z[\eta, \gamma] = \int D[\varphi, \sigma, \pi] \exp \{ iS[\varphi, \sigma, \pi] + is[\eta, \gamma] \}, \quad (19)$$

with action and source terms given by

$$S[\varphi, \sigma, \pi] = \int d^4x \left(i\varphi^\dagger D_+ \varphi + i\chi^\dagger D_- \chi - \frac{1}{2}(\sigma^2 + \pi^2) \right), \quad (20)$$

$$s[\eta, \gamma] = \int d^4x \left(\eta^\dagger \varphi + \varphi^\dagger \eta + \gamma^\dagger \chi + \chi^\dagger \gamma + \gamma^\dagger \frac{i}{D_-} \gamma \right). \quad (21)$$

Here the composite field χ is defined as

$$\chi = \chi[\varphi] = -\frac{1}{\sqrt{2}D_-} D_\perp \varphi. \quad (22)$$

In this formulation, the energy of a field configuration $\{\varphi, \sigma, \pi\}$ assumes a particularly simple form if expressed in terms of the composite field χ ,

$$E = \int d^3x \left(i\chi^\dagger D_- \chi + \frac{1}{2}(\sigma^2 + \pi^2) \right). \quad (23)$$

III. LOCAL MODULATIONS OF CHIRAL SYMMETRY TRANSFORMATIONS

A. Local Ward-Takahashi Identities

The process of eliminating the constrained degrees of freedom has not altered the symmetry properties of the system. In the chiral limit, the action $S[\varphi]$ [Eq. (13)] is invariant under the global transformation

$$\varphi \rightarrow e^{i\alpha\sigma_3} \varphi \quad (24)$$

since both $D_-[\varphi]$ and D_m are invariant and therefore

$$\chi \rightarrow e^{i\alpha\sigma_3} \chi.$$

Likewise the invariance of the action $S[\varphi, \sigma, \pi]$ [Eq. (20)] under the combined chiral rotation

$$\varphi \rightarrow e^{i\alpha\sigma_3} \varphi, \quad (25)$$

$$\begin{aligned} \sigma &\rightarrow \tilde{\sigma} = \sigma \cos 2\alpha + \pi \sin 2\alpha, \\ \pi &\rightarrow \tilde{\pi} = \pi \cos 2\alpha - \sigma \sin 2\alpha, \end{aligned} \quad (26)$$

is easily established. We also note that the expression (9) for the axial current remains valid, since the Lagrangian does not contain derivatives of the auxiliary fields.

The consequences of the presence of a symmetry, in particular the issue of spontaneous symmetry breaking, are in general studied by extending the global symmetry transformations characterized by α to local ones determined by a space-time dependent symmetry parameter $\alpha(x)$ [12]. In general, such transformations do not leave the Lagrangian invariant. The non-invariance is connected to the space-time derivatives of the fields in the Lagrangian. In ordinary coordinates with the local form of the Lagrangian the non-invariance can therefore be expressed in terms of space-time derivatives of the symmetry parameter and tends to zero when the wavelength of $\alpha(x)$ approaches infinity. It is therefore plausible that in local formulations of field theory the space-time modulated generator of the symmetry, *e.g.* the operator

$$U_5[\alpha] = \exp \left(i \int d^3x \psi^\dagger(x) \gamma_5 \alpha(x) \psi(x) \right) \quad (27)$$

in chirally symmetric theories, generates in the symmetry-broken case the soft modulations around the vacuum which can be interpreted as the associated NG particles. This method of local extension of the symmetry transformations is also the basis of the WT identities [12]. If for instance we carry out in the generating functional (11) the variable substitution

$$\varphi(x) \rightarrow e^{i\alpha(x)\sigma_3}\varphi(x), \quad \chi(x) \rightarrow e^{i\alpha(x)\sigma_3}\chi(x)$$

(replacing covariant derivatives by ordinary derivatives) the invariance of the path integral yields the functional identity

$$\begin{aligned} 0 = & \int D[\varphi, \chi] e^{i \int d^4x (\mathcal{L} + \eta^\dagger \varphi + \varphi^\dagger \eta + \gamma^\dagger \chi + \chi^\dagger \gamma)} \\ & \left[\partial_\mu j_5^\mu(x) + \sqrt{2}m (\varphi^\dagger \sigma_2 \chi + \chi^\dagger \sigma_2 \varphi)(x) \right. \\ & \left. + i (\eta^\dagger \sigma_3 \varphi - \varphi^\dagger \sigma_3 \eta + \gamma^\dagger \sigma_3 \chi - \chi^\dagger \sigma_3 \gamma)(x) \right]. \end{aligned} \quad (28)$$

By functional differentiation with respect to the sources the various chiral WT identities can be derived. For instance, application of

$$\Delta(y) = \sigma_2^{\beta\alpha} \left[\frac{\delta}{\delta \gamma_\alpha^\dagger(y)} \frac{\delta}{\delta \eta_\beta(y)} + \frac{\delta}{\delta \eta_\alpha^\dagger(y)} \frac{\delta}{\delta \gamma_\beta(y)} \right]_{\eta=\gamma=0} \quad (29)$$

yields the WT identity leading to the GOR relation

$$\begin{aligned} 0 = & \int D[\varphi, \chi] e^{i \int d^4x \mathcal{L}} \left[(\varphi^\dagger \sigma_2 \chi + \chi^\dagger \sigma_2 \varphi)(y) \right. \\ & \left. \left\{ \partial_\mu j_5^\mu(x) + \sqrt{2}m (\varphi^\dagger \sigma_2 \chi + \chi^\dagger \sigma_2 \varphi)(x) \right\} \right. \\ & \left. + 2i\delta(x-y) (\varphi^\dagger \sigma_1 \chi + \chi^\dagger \sigma_1 \varphi)(y) \right]. \end{aligned} \quad (30)$$

Although formulated in terms of the light-cone representation of the spinors, the derivation and the result are identical to the standard procedure. In light-cone quantization we deal only with the spinors φ as dynamical variables or, equivalently, as independent integration variables in the corresponding generating functional and derive WT identities by performing the variable substitution

$$\varphi(x) \rightarrow e^{i\alpha(x)\sigma_3}\varphi(x) \quad (31)$$

in the relevant path integrals such as (12). The induced transformation of the composite field χ [Eq. (15)] is non-local. Repeating the above procedure yields the following WT identity formulated in terms of the dynamical degrees of freedom φ

$$\begin{aligned} 0 = & \int D[\varphi] e^{i \int d^4x (\mathcal{L}_\varphi[\gamma, \gamma^\dagger] + \eta^\dagger \varphi + \varphi^\dagger \eta)} \left[\partial_\mu j_{5,\gamma}^\mu(x) \right. \\ & + \sqrt{2}m (\varphi^\dagger \sigma_2 \chi_\gamma + \chi_\gamma^\dagger \sigma_2 \varphi)(x) \\ & \left. + i (\eta^\dagger \sigma_3 \varphi - \varphi^\dagger \sigma_3 \eta + \gamma^\dagger \sigma_3 \chi_\gamma - \chi_\gamma^\dagger \sigma_3 \gamma)(x) \right], \end{aligned} \quad (32)$$

with

$$\chi_\gamma = \chi + \frac{i}{D_-} \gamma, \quad j_{5,\gamma} = j_5[\varphi, \varphi^\dagger, \chi_\gamma, \chi_\gamma^\dagger].$$

Integration over the constrained variables has led to a source dependent modification of the composite field which accounts for the fluctuations of χ . The WT identity for the φ variables is equivalent to the WT identity formulated in terms of the functional integral over φ and χ fields. In the present form, the WT identity has a canonical interpretation in light-cone quantization. GOR type relations can be derived from Eq. (32) by functional differentiation with respect to the sources.

B. Failure of Pion Dominance

Chiral WT identities are most useful if one considers their Fourier transform in the limit $p \rightarrow 0$. In normal coordinates this enables us to derive the well-known GOR relation for the pion mass away from the chiral limit [13] as well as to relate the pion Bethe-Salpeter (BS) amplitude to the quark propagator in the chiral limit [12]. Being dominated by symmetry aspects such relations have a universal character independent of the specific dynamics and are therefore of fundamental interest. The point $p = 0$ is singled out here since only at these kinematics a certain sum over many meson states appearing in the WT identity is saturated by the NG boson. In view of the notorious light-cone singularity at $p = 0$, it is not obvious whether the same procedure can be carried over to light-cone field theory. In this section we will show that this is indeed not the case: If one takes the limit $p_+, p_\perp \rightarrow 0$ keeping p_- finite, the sum over states is not dominated by the pion contribution but massive states can come in. We will also recall why this does not happen in ordinary coordinates. Once we have identified the source of the problem we will be able to cure it in later sections by an appropriate modification of the WT identities.

Our starting point is the WT identity (32) which is needed here for $\gamma = 0$ only. We apply the functional derivative

$$\delta_{\alpha\beta}(y, z) \equiv \frac{\delta^2}{\delta \eta_\alpha^\dagger(y) \delta \eta_\beta(z)} \Big|_{\eta=\gamma=0} \quad (33)$$

and obtain

$$\begin{aligned} 0 = & \int D[\varphi] e^{iS} \left\{ \left[\partial_\mu j_5^\mu(x) + \sqrt{2}m (\varphi^\dagger \sigma_2 \chi \right. \right. \\ & + \chi^\dagger \sigma_2 \varphi)(x) \left. \right] \varphi_\beta^\dagger(z) \varphi_\alpha(y) + \delta(x-y) \varphi_\beta^\dagger(z) (\sigma_3 \varphi)_\alpha(x) \\ & \left. - \delta(x-z) (\varphi^\dagger \sigma_3)_\beta(x) \varphi_\alpha(y) \right\}. \end{aligned} \quad (34)$$

The canonical form of Eq. (34) in the chiral limit is

$$\begin{aligned} \partial_\mu \langle 0 | T(j_5^\mu(x) \varphi_\beta^\dagger(z) \varphi_\alpha(y)) | 0 \rangle \\ = i [\delta(x-y) - \delta(x-z)] (\sigma_3)_{\alpha\beta} G(y-z), \end{aligned} \quad (35)$$

where G stands for the light-cone fermion propagator

$$\langle 0|T(\varphi_\alpha(y)\varphi_\beta^\dagger(z))|0\rangle = -i\delta_{\alpha\beta}G(y-z). \quad (36)$$

In the chiral limit the Fourier transform of Eq. (35), using

$$F_{5\alpha\beta}^\mu(p, q) \equiv \int d^4x d^4y e^{-i(px-qq)} \langle 0|T(j_5^\mu(x)\varphi_\beta^\dagger(0)\varphi_\alpha(y))|0\rangle, \quad (37)$$

assumes the compact form

$$p^\mu F_{5\alpha\beta}^\mu(p, q) = \sigma_3 (G(q-p) - G(q)). \quad (38)$$

To calculate the vertex function (37) by inserting a complete set of states, we choose the kinematical condition $p_- > q_- > 0$. Then, only the time ordering $x^+ < (y^+, 0)$ contributes and we obtain

$$p_\mu F_{5\alpha\beta}^\mu(p, q) = \sum_n \frac{p_\mu}{2p_-} \int dx^+ \int d^4y e^{-i(p_+ - p_+^{(n)})x^+ + iqy} \theta(y^+ - x^+) \theta(-x^+) \langle 0|T(\varphi_\beta^\dagger(0)\varphi_\alpha(y))|n; \vec{p}\rangle \langle n; \vec{p}|j_5^\mu(0)|0\rangle,$$

where $|n; \vec{p}\rangle$ is a mesonic state with mass m_n and the light-cone momenta are

$$\vec{p} = (p_-, \vec{p}_\perp), \quad p_+^{(n)} = \frac{m_n^2 + p_\perp^2}{2p_-}. \quad (39)$$

The integration over x^+ finally yields

$$p^\mu F_{5\alpha\beta}^\mu(p, q) = i \sum_n \frac{p_\mu \langle n; \vec{p}|j_5^\mu(0)|0\rangle}{p^2 - m_n^2 + i\epsilon} \int d^4y e^{iqy} \left\{ \theta(-y^+) \langle 0|\varphi_\beta^\dagger(-y)\varphi_\alpha(0)|n; \vec{p}\rangle e^{-ipy} - \theta(y^+) \langle 0|\varphi_\alpha(y)\varphi_\beta^\dagger(0)|n; \vec{p}\rangle \right\}. \quad (40)$$

Up to this point, the calculation in standard and in light-cone coordinates are essentially identical. Differences show up in the attempt to project out the contribution from the NG boson in the sum over n . To identify the origin of these differences, let us consider in some detail the expression $p_\mu \langle n; \vec{p}|j_5^\mu(0)|0\rangle$, treating ordinary and light-cone coordinates in parallel. Due to covariance, only pseudoscalar (PS) or axial vector (AV) states can contribute to the relevant current matrix element,

$$\langle n; \vec{p}|j_5^\mu(0)|0\rangle = \begin{cases} 2f_n p_\mu^{(n)} & \text{PS} \\ 2(f_n p_\mu^{(n)} + g_n \varepsilon_\mu) & \text{AV} \end{cases} \quad (41)$$

where ε is a polarization vector and $p^n = (\sqrt{m_n^2 + p_\perp^2}, \vec{p})$ replaces (39) in ordinary coordinates. Current conservation then implies

$$0 = p_\mu^{(n)} \langle n; \vec{p}|j_5^\mu(0)|0\rangle = \begin{cases} 2f_n m_n^2 & \text{PS} \\ 2(f_n m_n^2 + g_n \varepsilon p^{(n)}) & \text{AV} \end{cases} \quad (42)$$

From this we conclude that $f_n = 0$ for all massive PS states which therefore cannot contribute to the sum over n , independently of the coordinates chosen. For massive AV states, since $\varepsilon p^{(n)} = 0$, we can only infer that $f_n = 0$ whereas $g_n \neq 0$ in general, so that they could in principle contribute. The further discussion requires specification of the polarization vectors. In ordinary coordinates, the conditions $\varepsilon^2 = -1$ and $\varepsilon p^{(n)} = 0$ can be fulfilled by transverse or longitudinal polarization vectors as follows,

$$\begin{aligned} \varepsilon_t &= (0, \vec{n}_\perp) \\ \varepsilon_l &= \left(\frac{|\vec{p}|}{m_n}, \frac{p_0^{(n)}}{m_n} \frac{\vec{p}}{|\vec{p}|} \right) \end{aligned} \quad (43)$$

(\vec{n}_\perp is a unit vector orthogonal to \vec{p}). The light-cone components of ε are then defined in the usual way. The quantity which enters the WT identity (40) is $p_\mu \langle n; \vec{p}|j_5^\mu(0)|0\rangle$ which is in general non-vanishing not only for the massless PS but also for massive AV states. In ordinary coordinates, the unwanted AV states can be suppressed by going to the point $p_\mu \rightarrow 0$,

$$\begin{aligned} p_\mu \langle n; \vec{p}|j_5^\mu(0)|0\rangle &= (p_\mu - p_\mu^{(n)}) \langle n; \vec{p}|j_5^\mu(0)|0\rangle \\ &= (p_0 - p_0^{(n)}) 2g_n \varepsilon^0 \\ &\xrightarrow{p_\mu \rightarrow 0} 0 \quad \text{AV} \end{aligned} \quad (44)$$

We have used current conservation and the fact that $\varepsilon^0(\vec{p} = 0) = 0$, Eq. (43). In light-cone coordinates, we approach the point $p_+ = 0, p_\perp = 0$ keeping $p_- \neq 0$ and find

$$\begin{aligned} p_\mu \langle n; \vec{p}|j_5^\mu(0)|0\rangle &= (p_+ - p_+^{(n)}) \varepsilon^+ \\ &\xrightarrow{(p_+, \vec{p}_\perp) \rightarrow 0} -p_+^{(n)} \varepsilon^+ \neq 0 \quad \text{AV} \end{aligned} \quad (45)$$

since ε^+ does not vanish at this kinematical point. Thus the characteristic light-cone kinematics prevents dominance of the NG boson in the GOR type relation (35). More generally, this example demonstrates that indeed soft, local modulations of the form (31) do not excite exclusively massless states.

IV. NON-LOCAL MODULATIONS OF SYMMETRY TRANSFORMATIONS

A. Non-Local Ward-Takahashi Identities

Only in a local formulation of field theory, space-time modulated symmetry transformations as considered in Sec. III can be expected to give rise to small excitation energies for sufficiently long wavelengths of the modulation. In the non-local formulations obtained by integrating out the fields χ , a chiral transformation with a space-time dependent parameter $\alpha(x)$ cannot be expected to

generate excitations of low light-cone energy only. For illustration, we calculate the change in the energy (23) under the transformation (25,26) with a space-time dependent $\alpha(x)$. With

$$D_{\perp}[\tilde{\sigma}, \tilde{\pi}] = e^{i\alpha\sigma_3} D_{\perp}[\sigma, \pi] e^{-i\alpha\sigma_3} + (i\partial_1 - \sigma_3\partial_2)\alpha + im[\sigma_1(\cos 2\alpha - 1) - \sigma_2 \sin 2\alpha], \quad (46)$$

we find in the chiral limit and for infinitesimal, x_{\perp} independent $\alpha(x)$

$$\begin{aligned} \delta E &= i \int d^3x (\delta\chi^{\dagger} D_{-}\chi + \chi^{\dagger} D_{-}\delta\chi) \\ &= \int d^3x \varphi^{\dagger} D_{\perp}^{\dagger} \left[\frac{1}{D_{-}}, \alpha \right] \sigma_3 D_{\perp} \varphi. \end{aligned} \quad (47)$$

The change in energy induced by weakly modulated $\alpha(x)$ does not approach smoothly the limit $\delta E = 0$ for constant $\alpha(x)$. As a consequence we cannot expect excitations generated by the local extension of the symmetry transformations [Eq. (31)] to excite preferentially the NG bosons. Indeed we have seen in Sec. III B that GOR type of relations derived from the WT identity (32) are not saturated in the limit of vanishing momenta (p_{+}, p_{\perp}) by the NG bosons. The connection of large excitation energies with long-wavelength phenomena is similar to the Higgs mechanism where due to the long range nature of the Coulomb interaction gapless excitations are prevented altogether from appearing. Unlike in the case of the Higgs mechanism and despite the light-cone non-localities NG bosons must exist if the symmetry is spontaneously broken. Their representation in terms of local space-time modulated symmetry transformations of the basic fields φ is however lost. The expression for the change in energy (47) rather suggests to construct transformations which are local space-time modulated transformations of the composite field χ and not of the fundamental field φ .

Before we proceed with the explicit construction of such transformations in specific theories, we derive, independently of the particular dynamics, the general structure of WT identities following from the existence of massless particles. On the light-cone, formation of NG bosons is connected to the appearance of a new symmetry property. The (Goldstone) one-particle states with vanishing transverse momenta are degenerate with the ground state. As a consequence an operator $\Phi(x^{+}, x^{-})$, the field operator of the NG particles with vanishing transverse momentum, must exist with the property

$$[\Phi(x^{+}, x^{-}), H]|0\rangle = 0, \text{ i.e. }, \frac{\partial}{\partial x^{+}} \Phi(x^{+}, x^{-})|0\rangle = 0. \quad (48)$$

This equation can be rewritten formally as a WT identity

$$\begin{aligned} \frac{\partial}{\partial x^{+}} \langle 0|T(\Phi(x^{+}, x^{-})\mathcal{O}(y))|0\rangle = \\ \delta(x^{+} - y^{+}) \langle 0|[\Phi(x^{+}, x^{-}), \mathcal{O}(y)]|0\rangle, \end{aligned} \quad (49)$$

where $\mathcal{O}(y)$ is an operator local in light-cone time but otherwise arbitrary. In terms of a functional integral,

this WT identity reads

$$\begin{aligned} 0 &= \frac{\delta}{\delta\omega(y)} \int D[\varphi] e^{iS[\varphi]} \left[\frac{\partial}{\partial x^{+}} \Phi(x^{+}, x^{-}) \right. \\ &\quad \left. - \delta(x^{+} - y^{+}) (\Phi(x^{+} + \epsilon^{+}, x^{-}) - \Phi(x^{+} - \epsilon^{+}, x^{-})) \right]_{\epsilon=0} \end{aligned} \quad (50)$$

where

$$\frac{\delta}{\delta\omega(y)} e^{is} = \mathcal{O}(y) e^{is}$$

and the limit $\epsilon^{+} \rightarrow 0$ has to be taken after functional differentiation. We emphasize that the identities (49, 50) follow from the mere existence of massless particles. Their connection to symmetry properties is not specified though. It has to be established within a particular dynamical context.

We will first perform such an analysis for field theories like the NJL model, where the spontaneous breakdown of chiral symmetry occurs by fermion mass generation. In this case, the differential operator D_{\perp} [Eq. (18)] must contain the dynamically generated fermion mass M . In the presence of the four-fermion interaction this mass will be given in terms of the expectation values of the auxiliary fields

$$\langle \sigma \rangle = \frac{M}{g}, \quad \langle \pi \rangle = 0, \quad (51)$$

implying a non-vanishing chiral condensate $\langle 0|\bar{\psi}\psi|0\rangle$. At first sight, a condensate would seem to be incompatible with the triviality of the light-cone vacuum. In the context of various field-theoretic models, it has been demonstrated that this is not necessarily the case [14, 15]. If one defines expectation values of bilinear fermion operators more carefully via point-splitting in light-cone time, dynamical information about the specific theory comes in and can be shown to reproduce the same results as in ordinary coordinates. Equivalently, one should interpret operators which acquire vacuum expectation values (e.g. order parameters) canonically as equal x^{+} limits of Heisenberg operators rather than Schrödinger operators. Similar considerations apply whenever an operator is replaced by a c -number, for instance in the relation between D_{\perp} and D_M below.

To study the consequences of this assumed breakdown of the chiral symmetry, we consider the following non-local transformation of the fundamental fields $\varphi(x)$

$$\varphi(x) \rightarrow \tilde{\varphi}(x) = e^{i\alpha(x)\sigma_3} D_M^{-1} e^{-i\alpha(x)\sigma_3} D_M \varphi(x) \quad (52)$$

where D_M has been defined in Eq. (6). The transformation parameter $\alpha(x)$ is chosen to be independent of the transverse coordinates (x^1, x^2). In this case, the transformation becomes unitary and the Jacobian associated with the variable substitution (52) is one. The transformation (52) acts non-trivially in the broken phase when

the dynamical fermion mass M is non-vanishing, otherwise it reduces to unity. In the presence of the four-fermion interaction described by the auxiliary fields σ and π , the transformation (52) is always meant to be applied simultaneously with the rotation (26) of the auxiliary fields. The rationale behind this unfamiliar construction is the following: The dependent and independent spinors are related by

$$\chi = -\frac{1}{\sqrt{2}D_-}D_\perp\varphi. \quad (53)$$

The standard global chiral transformation (at $m = 0$) is

$$\begin{aligned} \varphi &\rightarrow \tilde{\varphi} = U\varphi, \\ D_\perp[\sigma, \pi] &\rightarrow D_\perp[\tilde{\sigma}, \tilde{\pi}] = e^{i\alpha}D_\perp[\sigma, \pi]e^{-i\alpha} \end{aligned} \quad (54)$$

with

$$U = e^{i\alpha} \quad (55)$$

and induces the transformation

$$\chi \rightarrow \tilde{\chi} = e^{i\alpha}\chi \quad (56)$$

of the dependent field χ . Here we modify this transformation such that for weakly (x^+, x^-) -dependent α the energy Eq. (23) remains unchanged. This is achieved with the following choice of U ,

$$U = e^{i\alpha\sigma_3} \frac{1}{D_\perp} e^{-i\alpha\sigma_3} D_\perp. \quad (57)$$

By construction both χ in Eq. (53) and $\sigma^2 + \pi^2$ are invariant. The transformation (52) follows if we approximately identify D_\perp with D_M .

Returning to the transformation (52), the definition of the composite field χ , Eq. (15), then entails the following transformation

$$\begin{aligned} \chi \rightarrow \tilde{\chi} &= -\frac{1}{\sqrt{2}D_-}D_\perp[\tilde{\sigma}, \tilde{\pi}]\tilde{\varphi} \\ &= \frac{1}{D_-} \left(e^{i\alpha\sigma_3} \Omega e^{-i\alpha\sigma_3} \Omega^{-1} D_- \chi \right. \\ &\quad \left. - \frac{im}{\sqrt{2}} [\sigma_1(\cos 2\alpha - 1) - \sigma_2 \sin 2\alpha] \tilde{\varphi} \right) \end{aligned} \quad (58)$$

where we have defined

$$\Omega[\sigma, \pi] = D_\perp[\sigma, \pi] D_M^{-1}. \quad (59)$$

We now proceed by neglecting the fluctuations of the auxiliary fields σ and π around their vacuum expectation values (51), *i.e.*, we replace these fields by their stationary point in the functional integral. In this approximation we can identify the differential operators D_\perp [Eq. (18)] and D_M [Eq. (6)], so that $\Omega[\sigma, \pi] \rightarrow 1$. In the chiral limit, χ then remains invariant [cf. Eq. (58)] while the elementary field φ transforms non-trivially in the broken

phase where σ develops a finite expectation value M . As pointed out above also the energy of a field configuration [Eq. (23)] remains invariant under the x^- -dependent symmetry transformations and therefore the transformation defined by Eqs. (52) and (26) generates massless excitations with arbitrary values of the momentum component p_- .

The same conclusion may also be reached by considering the WT identity associated with the transformation (52). To this end, we consider the infinitesimal version of the transformation (26) and (52) with x_\perp -independent α . We then obtain

$$\delta\varphi(x) = i\alpha(x^+, x^-)\Sigma_3\varphi(x) \quad (60)$$

with

$$\Sigma_3 = D_M^{-1}[\sigma_3, D_M] = \frac{2M^2\sigma_3 - 2iM\vec{\sigma}_\perp\vec{\partial}_\perp}{-\Delta_\perp + M^2}. \quad (61)$$

The operator Σ_3 coincides with the pion operator at zero transverse momentum which can be derived in the NJL model by solving the light-cone BS equation explicitly in ladder approximation, see the Appendix. To leading order in the expansion of Ω [Eq. (59)], χ then transforms only in the presence of a non-vanishing bare quark mass,

$$\delta\chi = i\alpha(x^+, x^-)\frac{\sqrt{2}m}{D_-}\sigma_2\varphi. \quad (62)$$

Performing this variable substitution in the expression (20) for the action, we obtain

$$\begin{aligned} 0 &= \int D[\varphi] e^{iS[\varphi, \sigma, \pi] + iS[\eta, \gamma]} \int d^2x_\perp \\ &\quad \left[(D_+^* \varphi^\dagger(x) + i\eta^\dagger(x)) (\Sigma_3\varphi)(x) \right. \\ &\quad \left. + \sqrt{2}m(\chi^\dagger\sigma_2\varphi)(x) + \text{c.c.} \right] \end{aligned} \quad (63)$$

Applying the differential operator $\Delta(y)$ of Eq. (29) we obtain another GOR like relation

$$\begin{aligned} 0 &= \int D[\varphi] e^{iS[\varphi, \sigma, \pi]} \left[\int d^2x_\perp (\varphi^\dagger\sigma_2\chi + \chi^\dagger\sigma_2\varphi)(y) \right. \\ &\quad \left\{ \partial_+(\varphi^\dagger\Sigma_3\varphi)(x) + \sqrt{2}m(\chi^\dagger\sigma_2\varphi + \varphi^\dagger\sigma_2\chi)(x) \right\} \\ &\quad \left. - \delta(x^+ - y^+)\delta(x^- - y^-) (\varphi^\dagger\Sigma_3\sigma_2\chi - \chi^\dagger\sigma_2\Sigma_3\varphi)(y) \right]. \end{aligned} \quad (64)$$

We recognize in this expression for the chiral limit the structure of the general WT identity (50) derived from the existence of NG bosons. We have succeeded in connecting the meson field operator with the generator of a unitary x^\pm -dependent transformation

$$\Phi(x^+, x^-) = \int d^2x_\perp \varphi^\dagger \Sigma_3 \varphi \quad (65)$$

and, in the chiral limit, Eq. (50) coincides with the WT identity (64) for the choice

$$\mathcal{O} = \varphi^\dagger \sigma_2 \chi + \chi^\dagger \sigma_2 \varphi.$$

In general Eqs. (48) and (49) do not hold as operator identities. Interactions between the NG particles as considered in the corrections to the leading order term [see Eq. (59)] prevent the vanishing of the commutator in the whole Hilbert space.

The operator Σ_3 plays a prominent role here, both as pion operator and as the quantity which replaces σ_3 when going from the generator of local to the one of non-local chiral transformations on the light-cone. Actually, one can understand the structure and role of Σ_3 rather simply on a purely classical level. Consider axial current conservation integrated over x_\perp ,

$$\int d^2 x_\perp \partial_\mu j_5^\mu = \int d^2 x_\perp [\partial_+ (\varphi^\dagger \sigma_3 \varphi) + \partial_- (\chi^\dagger \sigma_3 \chi)] = 0. \quad (66)$$

Using the Euler-Lagrange equations

$$\sqrt{2} \partial_+ \varphi = -D_\perp^\dagger \chi, \quad \sqrt{2} \partial_- \chi = -D_\perp \varphi, \quad (67)$$

the ∂_- term in Eq. (66) can be transformed with the help of

$$\begin{aligned} \partial_- (\chi^\dagger \sigma_3 \chi) &= \varphi^\dagger \left(D_\perp^\dagger \sigma_3 \frac{1}{D_\perp^\dagger} \right) \partial_+ \varphi \\ &+ (\partial_+ \varphi)^\dagger \left(\frac{1}{D_\perp} \sigma_3 D_\perp \right) \varphi. \end{aligned} \quad (68)$$

In the approximation $D_\perp \rightarrow D_M$ used above, we have

$$D_\perp^\dagger \sigma_3 \frac{1}{D_\perp^\dagger} = \frac{1}{D_\perp} \sigma_3 D_\perp \rightarrow \frac{D_M^\dagger \sigma_3 D_M}{D_M^\dagger D_M} \quad (69)$$

independent of x^+ so that the x_\perp -integrated axial current conservation goes over into a local (x^- -dependent) conservation law,

$$\partial_+ \int d^2 x_\perp \varphi^\dagger \Sigma_3 \varphi = 0. \quad (70)$$

We thus reproduce Eqs. (48) and (65) in the classical limit. It is instructive to perform the analogous calculation in 1+1 dimensions as well. In a chiral representation ($\gamma^0 = \sigma_1, \gamma^1 = -i\sigma_2, \gamma_5 = \sigma_3$), vector and axial vector current conservation assume the form

$$\begin{aligned} \partial_\mu j^\mu &= \partial_+ (\varphi^\dagger \varphi) + \partial_- (\chi^\dagger \chi) = 0, \\ \partial_\mu j_5^\mu &= \partial_+ (\varphi^\dagger \varphi) - \partial_- (\chi^\dagger \chi) = 0. \end{aligned} \quad (71)$$

Adding these two equations we get a local conserved quantity, the pion operator, independently of any dynamics,

$$\partial_+ (\varphi^\dagger \varphi) = 0. \quad (72)$$

The relation between the appearance of massless particles and a local symmetry in the light-cone approach to 1+1 dimensional field theories has been noted before [16]. In the case of the Schwinger model [17], the axial anomaly generates a mass term, see Sec. IV B.

To analyze further the chiral WT identity (64) we proceed as usual by Fourier-transforming the two point functions

$$G_\Sigma^+(p) = \int d^4 x e^{ipx} \langle 0 | T(\varphi^\dagger(x) \Sigma^3 \varphi(x) (\varphi^\dagger(0) \sigma_2 \chi(0) + \text{c.c.})) | 0 \rangle,$$

$$G_5(p) = \int d^4 x e^{ipx} \langle 0 | T((\chi^\dagger(x) \sigma_2 \varphi(x) + \text{c.c.}) (\varphi^\dagger(0) \sigma_2 \chi(0) + \text{c.c.})) | 0 \rangle$$

and defining the condensate [cf. Eq. (9)]

$$\Gamma = \frac{i}{2\sqrt{2}} \langle 0 | \varphi^\dagger(y) \Sigma_3 \sigma_2 \chi(y) - \chi^\dagger(y) \sigma_2 \Sigma_3 \varphi(y) | 0 \rangle.$$

The WT identity becomes

$$p_+ G_\Sigma^+(p_+, p_-, \vec{0}) - i\sqrt{2} m G_5(p_+, p_-, \vec{0}) = -i\sqrt{2} \Gamma. \quad (73)$$

Expressing G_Σ^+ and G_5 as sums of contributions from mesonic states as we did in Sec. III B, we have

$$\sum_n \frac{p_+ B_n(p_-) C_n(p_-) + i\sqrt{2} m |C_n(p_-)|^2}{p_+ - \frac{m_n^2}{2p_-} + i \text{sgn}(p_-) \epsilon} = -i\sqrt{2} \Gamma, \quad (74)$$

where B_n and C_n are defined by

$$\begin{aligned} B_n(p_-) &\equiv \langle n; p_- | \varphi^\dagger \Sigma_3 \varphi | 0 \rangle, \\ C_n(p_-) &\equiv \langle 0 | (\chi^\dagger \sigma_2 \varphi + \text{c.c.}) | n; p_- \rangle. \end{aligned} \quad (75)$$

Examining the above expression near $p_+ = 0$ in the chiral limit, we see that a massless meson (pion, $m_\pi = 0$) must exist for non-vanishing condensate (Γ) and obtain

$$B_\pi(p_-) C_\pi(p_-) = -i\sqrt{2} \Gamma. \quad (76)$$

B_π is related to the pion decay constant f_π defined in Eq. (41) as $B_\pi(p_-) = 4p_- f_\pi$. For small m , near the pion pole, we have

$$m_\pi^2 B_\pi(p_-) = i2\sqrt{2} p_- C_\pi^*(p_-) \quad (77)$$

which is nothing but the GOR relation

$$m_\pi^2 f_\pi^2 = m \Gamma = -m \langle \bar{\psi} \psi \rangle. \quad (78)$$

We have thus shown for the NJL model that the modified non-local chiral transformation which leaves the composite field χ invariant in the chiral limit generates useful WT identities. Unlike in Sec. III B, here pion dominance holds on the light-cone, leading correctly to the massless NG boson and the GOR relation.

B. Quark Propagator and Pion Structure Function

In ordinary coordinates WT identities have been used to derive a general relation between the BS amplitude of the pion at total momentum zero on the one hand and the quark propagator on the other hand [12]. It is interesting to repeat this derivation in the light-cone approach where the BS amplitude is closely related to the structure function of the pion. In the context of strong interaction physics, this opens the possibility to relate quark propagator and pion structure function. In ordinary coordinates where the BS amplitude is closely related to the form factor of the pion rather than to the structure function such a connection is much more remote.

Equipped with the tool of “non-local WT identities” developed in Sec. IV A, we now use this technique to derive the desired relationship. Applying $\delta_{\alpha\beta}(y, z)$ defined in Eq. (33) to the identity (63) in the chiral limit yields

$$0 = \int D[\varphi] e^{iS[\varphi, \sigma, \pi]} \quad (79)$$

$$\left\{ \int d^2x_\perp \partial_+ (\varphi^\dagger(x) \Sigma_3 \varphi(x)) \varphi_\alpha(y) \varphi_\beta^\dagger(z) \right. \\ \left. + \delta(x^+ - y^+) \delta(x^- - y^-) (\Sigma_3 \varphi)_\alpha(y) \varphi_\beta^\dagger(z) \right. \\ \left. - \delta(x^+ - z^+) \delta(x^- - z^-) \varphi_\alpha(y) (\Sigma_3^* \varphi^\dagger)_\beta(z) \right\}$$

or equivalently the WT identity

$$\partial_+ \int d^2x_\perp \langle 0 | T(\varphi^\dagger(x) \Sigma_3 \varphi(x)) \varphi_\alpha(y) \varphi_\beta^\dagger(z) | 0 \rangle = \quad (80)$$

$$- \delta(x^+ - y^+) \delta(x^- - y^-) \langle 0 | T(\Sigma_3 \varphi)_\alpha(y) \varphi_\beta^\dagger(z) | 0 \rangle \\ + \delta(x^+ - z^+) \delta(x^- - z^-) \langle 0 | T \varphi_\alpha(y) (\Sigma_3^* \varphi^\dagger)_\beta(z) | 0 \rangle.$$

With the definition of the three-point function

$$F_{\alpha\beta}(y - x, z - x) = \langle 0 | T(\varphi^\dagger(x) \Sigma_3 \varphi(x)) \varphi_\alpha(y) \varphi_\beta^\dagger(z) | 0 \rangle, \quad (81)$$

the WT identity at $x^+ = x^- = 0$ reads

$$\partial_+ \int d^2x_\perp F_{\alpha\beta}(y - x, z - x) \Big|_{x^+=0} = \\ - \delta(y^+) \delta(y^-) \int d^4\xi (y | (\Sigma_3)_{\alpha\beta} | \xi) G(\xi - z) \\ + \delta(z^+) \delta(z^-) \int d^4\xi G(y - \xi) (z | (\Sigma_3^*)_{\beta\alpha} | \xi), \quad (82)$$

with the fermion propagator defined in Eq. (36). In terms of the Fourier transformed three-point function

$$F(p, q) = \int d^4y d^4z e^{iqy + i(p-q)z} F(y, z), \quad (83)$$

the WT identity in momentum space becomes

$$p_+ F(p, q) = \Sigma_3(q) \theta(q_-) G(q) - \Sigma_3(q-p) \theta(q_- - p_-) G(q-p). \quad (84)$$

The general form of the fermion propagator (in manifestly covariant theories) is

$$G(q) = \frac{2q_- A}{q^2 A^2 - B^2} = \frac{A}{(q_+ - q_\perp^2/2q_-) A^2 - B^2/2q_-}, \quad (85)$$

where A and B are still arbitrary functions of $q^2 = 2q_+ q_- - q_\perp^2$. $\Sigma_3(q)$ is defined by the Fourier transform

$$\Sigma_3(q) = \int d^4x e^{-iqx} \langle 0 | \Sigma_3 | x \rangle = \frac{2M}{\sigma_\perp q_\perp + M \sigma_3}, \quad (86)$$

such that for $p_\perp = 0$, $\Sigma_3(q-p) = \Sigma_3(q)$. The one-pion pole term of $F(p, q)$ is given by

$$[F(p, q)]_{\text{pole}} = 2f_\pi \frac{1}{p_+} \Psi(p, q), \quad (87)$$

where we have used the relation between $B_\pi(p_-)$ and f_π . $\Psi(p, q)$ in (87) is the Fourier transform of the BS amplitude

$$\Psi_{\alpha\beta}(x, y) = \sqrt{2p_-} \langle 0 | T \varphi_\alpha(x) \varphi_\beta^\dagger(y) | \pi; p \rangle. \quad (88)$$

In the one-pion pole approximation and at $p_\perp = 0$, the BS amplitude is completely determined by the fermion propagator

$$\Psi(p, q) = \frac{1}{2f_\pi} \Sigma_3(q) [\theta(q_-) G(q) - \theta(q_- - p_-) G(q-p)]. \quad (89)$$

The light-cone wave function for the pion is given by the BS amplitude as

$$\Pi(p_-, q_-, q_\perp) = \int dq_+ \Psi(p, q), \quad (90)$$

and in turn is related to the pion structure function as

$$\mathcal{F}\left(\frac{q_-}{p_-}\right) = \int d^2q_\perp |\Pi(p_-, q_-, q_\perp)|^2. \quad (91)$$

Let us briefly indicate the modifications necessary for the discussion of 1+1 dimensional field theories below. For the Gross-Neveu model [7] written in terms of auxiliary fields σ and π and with the constrained variable χ integrated out, the action is of the same form as that of the NJL model if the composite field χ is defined as

$$\chi = \chi[\varphi] = \frac{1}{\sqrt{2i}\partial_-} (m + g\Sigma^\dagger) \varphi, \quad \Sigma = \sigma + i\pi. \quad (92)$$

For modulations of the fields in the chiral limit

$$\delta\varphi = i\alpha(x^+, x^-) \varphi, \quad \delta\Sigma = i\alpha(x^+, x^-) \Sigma, \quad \delta\chi = 0 \quad (93)$$

we are provided with

$$0 = \int D[\varphi] e^{iS[\varphi, \sigma, \pi] + iS[\eta, \gamma]} \{ (\partial_+ \varphi^\dagger(x) + i\eta^\dagger(x)) \varphi(x) + \text{c.c.} \}. \quad (94)$$

For the 't Hooft model [8] on the other hand, the Lagrangian in the $A_- = 0$ gauge is given by

$$\mathcal{L} = \varphi_i^\dagger (i\partial_+ \delta_{ij} + eA_{+ij}) \varphi_j + i\chi_i^\dagger \partial_- \chi_i + \partial_- A_{+ij} \partial_- A_{+ji} - \frac{m}{\sqrt{2}} (\varphi_i^\dagger \chi_i + \chi_i^\dagger \varphi_i). \quad (95)$$

A_{+ij} is the solution of Poisson's equation,

$$A_{+ij} = \frac{e}{2} \frac{1}{\partial_-^2} \varphi_j^\dagger \varphi_i. \quad (96)$$

In the chiral limit, the χ_i are zero and for modulations of the fields

$$\delta\varphi_i = i\alpha(x^+, x^-) \varphi_i, \quad (97)$$

we obtain once again the same WT identity. The relation between the BS amplitude and the fermion propagator can be taken over from the (3+1)-dimensional case by simply setting $\Sigma_3 = 1$.

We now apply these formulae to concrete models, starting from Eq. (89). Consider the NJL model first. To leading order in the $1/N$ expansion, the Fermion propagator is just a free, massive Green's function (dynamical mass M),

$$G(q) = \left(q_+ - \frac{q_\perp^2 + M^2}{2q_-} + i \operatorname{sgn}(q_-) \epsilon \right)^{-1}. \quad (98)$$

Integration over the relative energy variable,

$$\int dq_+ G(q) = -i\pi \operatorname{sgn}(q_-), \quad (99)$$

yields at once the following BS amplitude for the pion,

$$\int dq_+ \Psi(p_+, p_-, q) = -\frac{i\pi}{2f_\pi} \Sigma_3(q_\perp) [\theta(q_-) - \theta(q_- - p_-)], \quad (100)$$

in agreement with the result of the explicit solution of the model on the light-cone (see the Appendix). Incidentally, had we used the "local WT identity" of Secs. III A and III B, the result for the pion BS amplitude would differ from Eq. (100) by the replacement $\Sigma_3 \rightarrow \sigma_3$ and hence give the wrong answer. As explained in Sec. III B this discrepancy is due to the fact that axial vector states spoil pion dominance in the light-cone calculation. In the case of the NJL model, one can actually verify explicitly that the sum over axial vector states accounts for the difference between Σ_3 and σ_3 by making use of the $q\bar{q}$ scattering state solutions of the light-cone Tamm-Dancoff equation.

For the chiral Gross-Neveu model, the calculation is essentially the same [drop q_\perp^2 in the propagator, Eq. (98)]. Since $\Sigma_3 = 1$ in this case, the light-cone wavefunction reduces to the difference of step functions and becomes constant in the interval $x = p_-/P_- \in [0, 1]$.

In the case of the 't Hooft model, the quark propagator does not have the standard covariant form. One finds [8]

$$G(q) = \left(q_+ + \frac{Ng^2}{2\pi} \frac{1}{2q_-} - \frac{Ng^2}{2\pi\lambda} \operatorname{sgn}(q_-) + i \operatorname{sgn}(q_-) \epsilon \right)^{-1} \quad (101)$$

where the limit $\lambda \rightarrow 0$ moves the pole to infinity. If we perform the q_+ integration before taking the limit $\lambda \rightarrow 0$, the result is the same as in the chiral Gross-Neveu model. Note that in our derivation, we have assumed a covariant quark propagator, so that the final formula strictly speaking cannot be applied to gauge theories. Nevertheless, the result seems to be more general than the derivation. The pion wavefunction on the light-cone does not reflect the drastic difference in the fermion propagators of a confining and a free massive theory, so that quite some information seems to get lost when going from the propagator to the structure function due to the integration over q_+ . The simple results for 't Hooft and Gross-Neveu model pion wave functions agree with the literature [8, 18].

As a last application of our formalism we consider the Schwinger model [17], paying special attention to the axial anomaly on the light-cone. The Lagrangian density in the $A_- = 0$ gauge is

$$\mathcal{L} = \varphi^\dagger (i\partial_+ + eA_+) \varphi + \frac{1}{2} (\partial_- A_+)^2. \quad (102)$$

To obtain the WT identities with the modulation of the fields

$$\delta\varphi(x) = i\alpha(x) \varphi(x), \quad (103)$$

we have to consider the variations in the measures of the functional integral

$$\begin{aligned} \delta D[\varphi] &= i \int d^2x \alpha(x) J(x) D[\varphi], \\ \delta D[\varphi^\dagger] &= -i \int d^2x \alpha(x) \tilde{J}(x) D[\varphi^\dagger]. \end{aligned} \quad (104)$$

J and \tilde{J} are divergent quantities and we regularize them with the square of the Dirac operator, in analogy with Fujikawa's method [19] in ordinary (Euclidean) coordinates. The Dirac operator in the present model before eliminating the non-dynamical (vanishing) component of the fermion is a 2×2 matrix

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_- \\ \partial_+ - ieA_+ & 0 \end{pmatrix}, \quad (105)$$

and thus \mathcal{D}^2 is a diagonal matrix with the upper and lower elements, \mathcal{D}_+^2 and \mathcal{D}_-^2 respectively, given by

$$\mathcal{D}_+^2 = \partial_- (\partial_+ - ieA_+), \quad \mathcal{D}_-^2 = (\partial_+ - ieA_+) \partial_-. \quad (106)$$

Since φ is the upper component, J is regularized by the upper diagonal element as

$$J(x) = \langle x | \exp \left\{ i \frac{\mathcal{D}_+^2}{\Lambda^2} \right\} | x \rangle = \frac{\Lambda^2}{2\pi} \left(1 + \frac{e}{2\Lambda^2} \partial_- A_+(x) \right). \quad (107)$$

In accord with the covariant measure $D[\psi]D[\bar{\psi}]$ used in ordinary coordinates, φ^\dagger should be considered as the lower component in the present representation and \tilde{J} is to be regularized by the hermitian conjugate of the lower diagonal element which is equal to the upper element. Thus we obtain

$$\tilde{J}(x) = (x | \exp \left\{ -i \frac{p_+^2}{\Lambda^2} \right\} | x) = \frac{\Lambda^2}{2\pi} \left(1 - \frac{e}{2\Lambda^2} \partial_- A_+(x) \right). \quad (108)$$

The WT identity obtained by the above modulation can be expressed as

$$\partial_+ \varphi^\dagger(x) \varphi(x) = -J(x) + \tilde{J}(x) = -\frac{e}{2\pi} \partial_- A_+(x), \quad (109)$$

which represents the anomaly in the Schwinger model. Eliminating the gauge field, A_+ , by resolving the Poisson equation

$$\partial_-^2 A_+ = e \varphi^\dagger \varphi, \quad (110)$$

we finally obtain

$$2\partial_- \partial_+ \varphi^\dagger \varphi = -\frac{e^2}{\pi} \varphi^\dagger \varphi, \quad (111)$$

the Klein-Gordon equation of the Schwinger particle with mass $e/\sqrt{\pi}$.

V. CONCLUSIONS

We have studied chiral symmetry in the framework of light-cone field theory. We have analyzed the properties of the Nambu-Goldstone realization of the chiral symmetry by formulating appropriate WT identities. The non-local form of the light-cone Hamiltonian makes a revision of the standard analysis necessary. We have investigated the general structure of the chiral WT identities and have carried out explicit constructions for the NJL model, the Gross-Neveu model, two dimensional QCD and the Schwinger model. Basic elements in our construction are independent of the specific dynamics. In particular we have demonstrated that derivation of the relevant identities requires consideration of symmetry transformations which in general cannot be generated by soft space-time dependent modulations of the underlying global chiral rotations. Our results are a first step towards formulation of the Goldstone theorem in light-cone field theory. As we have shown, in light cone field theory, the existence of massless particles leads to a WT identity without reference to a spontaneous symmetry breakdown. The connection however of the creation operator of the NG particle with symmetry transformations could be established only within a specific dynamical context. In particular in the case of the NJL model our detailed construction assumes that fermion mass generation is the basic mechanism for the spontaneous chiral symmetry breakdown. Application of WT identities leads, like in ordinary coordinates, to a series of relations connecting properties of

the NG bosons with properties of the fermionic degrees of freedom. The GOR relation has the same structure and physical content as the corresponding relation formulated in ordinary coordinates. A novel relation between the fermion-propagator and the structure function of the NG bosons has been deduced from the light-cone WT identity.

An important next step in the study of chiral symmetry breakdown will be the application of the techniques developed here to gauge theories. Investigation of the chiral symmetry breakdown in QED by an external magnetic field should be a comparatively straightforward extension of the present investigation. Similarly, the extension of our light-cone calculation of the axial anomaly in 1+1 dimensions to 3+1 dimensions should be examined. A study of chiral WT identities in light-cone QCD may provide a new perspective on the mechanism driving the spontaneous chiral symmetry breakdown. Judging from our treatment of the 't Hooft model, fermion mass generation may not be the relevant process. Irrespective of their detailed structure, these WT identities will among others lead to a relation between the pion structure function and the quark propagator, which could be of considerable phenomenological interest.

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Appendix: NJL Model on the Light-Cone in Tamm-Dancoff Approximation

We summarize here the canonical formulation of the (one flavor) NJL model on the light-cone (see also [20, 21, 22]). The action with the auxiliary fields σ and π integrated out can be expressed as

$$S[\varphi] = \int d^4x \left(i\varphi^\dagger D_+ \varphi + i\chi^\dagger \partial_- \chi - \frac{1}{2}(\sigma^2 + \pi^2) \right), \quad (112)$$

where χ is given in Eq. (22), and σ and π are now the solutions of the following constraint equations,

$$\sigma + \frac{g}{2} \varphi^\dagger \left(\sigma_1 \frac{1}{\partial_-} D_\perp - D_\perp^\dagger \frac{1}{\partial_-} \sigma_1 \right) \varphi = 0, \quad (113)$$

$$\pi + \frac{g}{2} \varphi^\dagger \left(\sigma_2 \frac{1}{\partial_-} D_\perp - D_\perp^\dagger \frac{1}{\partial_-} \sigma_2 \right) \varphi = 0. \quad (114)$$

Eqs. (113) and (114) can be solved for σ and π in a formal $1/N$ expansion by writing

$$\sigma(x) \approx \sigma^{(0)} + \sigma^{(1)}(x), \quad \pi(x) \approx \sigma^{(2)}(x), \quad (115)$$

where $\sigma^{(0)}$ is the dominant c -number part while $\sigma^{(1,2)}$ describe the fluctuations in the σ ($0^+, n = 1$) and π ($0^-, n = 2$) channels. To leading order in this expansion, one can replace D_\perp by D_M , cf. Eq. (6). The c -number part $\sigma^{(0)}$ is determined self-consistently via the gap equation

$$\sigma^{(0)} = \frac{1}{2}g\langle 0|\varphi^\dagger\sigma_1\chi + \chi^\dagger\sigma_1\varphi|0\rangle, \quad (116)$$

whereas the fluctuations satisfy

$$\begin{aligned} \sigma^{(n)}(\vec{x}) + \int d^3y F(\vec{x} - \vec{y})\sigma^{(n)}(\vec{y}) = \\ -\frac{g}{2}\int d^3y \varphi^\dagger(\vec{x})\sigma_n(\vec{x})\left|\frac{1}{\partial_-}\right|\vec{y})D_M\varphi(\vec{y}) + \text{h.c.} \end{aligned} \quad (117)$$

Here, the kernel F is defined as

$$\begin{aligned} F(\vec{x} - \vec{y}) \equiv & -\frac{i}{2}g^2 \left(\langle \varphi^\dagger(\vec{x})\varphi(\vec{y}) \rangle (\vec{x}) \left| \frac{1}{\partial_-} \right| \vec{y} \right) \\ & + \langle \varphi^\dagger(\vec{y})\varphi(\vec{x}) \rangle (\vec{y}) \left| \frac{1}{\partial_-} \right| \vec{x} \rangle. \end{aligned} \quad (118)$$

Using these equations together with the normal mode expansion of φ ,

$$\varphi_\alpha(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} e^{-iqx} b_\alpha(\vec{q}), \quad (119)$$

($b_\alpha(\vec{q})$ annihilates (creates) a fermion (anti-fermion) for positive (negative) q_-), the Hamiltonian up to 2nd order in $1/\sqrt{N}$ becomes

$$\begin{aligned} H = & \int \frac{d^3q}{(2\pi)^3} \frac{q_\perp^2 + M^2}{2q_-} b_\alpha^\dagger(\vec{q}) b_\alpha(\vec{q}) \\ & - \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} (1 + F(p_-)) \sum_{n=1,2} \sigma^{(n)}(\vec{p}) \sigma^{(n)}(-\vec{p}), \end{aligned} \quad (120)$$

i.e., a sum of a free, massive Hamiltonian and a 2-term separable potential. The formfactors of the interaction term are

$$\begin{aligned} \sigma^{(n)}(\vec{p}) = & \frac{1}{1 + F(p_-)} \left(-\frac{g}{2} \right) \\ & \int \frac{d^3q}{(2\pi)^3} \mathcal{V}_{\alpha\beta}^{(n)}(\vec{p}, \vec{q}) b_\alpha^\dagger(\vec{q}) b_\beta(\vec{q} - \vec{p}), \end{aligned} \quad (121)$$

where

$$\begin{aligned} \mathcal{V}^{(n)}(\vec{p}, \vec{q}) = & [\sigma_n(-(\not{q}_\perp - \not{p}_\perp) + \sigma_1 M)] \frac{1}{q_- - p_-} \\ & + [(-\not{q}_\perp^* + \sigma_1 M)\sigma_n] \frac{1}{q_-} \end{aligned} \quad (122)$$

with $\not{q}_\perp = \sigma_3 q_1 + iq_2$ etc. In Tamm-Dancoff approximation, the meson state vector is expanded as

$$|\sigma^{(n)}; \vec{p}\rangle = \int \frac{d^3q}{(2\pi)^3} S_{\alpha\beta}^{(n)}(\vec{p}, \vec{q}) b_\alpha^\dagger(\vec{q}) b_\beta(\vec{q} - \vec{p}) |0\rangle. \quad (123)$$

The Schrödinger equation within the particle-anti-particle subspace,

$$\langle 0|b_\beta^\dagger(\vec{q} - \vec{p}) b_\alpha(\vec{q})(E - H)|\sigma^{(n)}; \vec{p}\rangle = 0, \quad (124)$$

then gives the Tamm-Dancoff equation

$$\begin{aligned} \left(E - \frac{M^2 + q_\perp^2}{2q_-} - \frac{M^2 + q_\perp^2}{2(p_- - q_-)} \right) S^{(n)}(\vec{p}, \vec{q}) \\ = \frac{1}{1 + F(q_-)} \left(-\frac{g^2}{2} \right) \mathcal{V}^{(n)}(\vec{p}, \vec{q}) \\ \int \frac{d^3q'}{(2\pi)^3} \text{tr} \mathcal{V}^{(n)}(-\vec{p}, \vec{q}' - \vec{p}) S^{(n)}(\vec{p}, \vec{q}'). \end{aligned} \quad (125)$$

Since the interaction term on the r.h.s. is separable, the eigenvalue problem reduces to an algebraic equation which for $p_\perp = 0$ reads

$$\begin{aligned} 0 = 1 + \frac{1}{1 + \tilde{F}(p_-)} \frac{g^2}{2} \int \frac{d^3q}{(2\pi)^3} \frac{\theta(q_-)\theta(p_- - q_-)}{E - \frac{M^2 + q_\perp^2}{2q_-} - \frac{M^2 + q_\perp^2}{2(p_- - q_-)}} \\ \left[(q_\perp^2 + M^2) \left(\frac{1}{q_-} + \frac{1}{p_- - q_-} \right)^2 - \frac{\mu_n^2}{q_-(p_- - q_-)} \right] \end{aligned} \quad (126)$$

with $\mu_1 = 2M$ and $\mu_2 = 0$. Using the gap equation together with Eq. (126), one finds two bound states with eigenvalues $\mu_n^2/(2p_-)$. The particle-anti-particle amplitude $S^{(n)}$ of each eigenstate is given by

$$S^{(n)}(\vec{p}, \vec{q}) = N^{(n)}(\vec{p}) \frac{\theta(q_-)\theta(p_- - q_-)}{E - \frac{M^2 + q_\perp^2}{2q_-} - \frac{M^2 + q_\perp^2}{2(p_- - q_-)}} \mathcal{V}^{(n)}(\vec{p}, \vec{q}) \quad (127)$$

where $N^{(n)}(\vec{p})$ is a normalization factor. For the pion in the chiral limit, the amplitude takes a particularly simple form,

$$S^{(2)}(\vec{p}, \vec{q}) = -2iN^{(2)}(\vec{p}) \frac{\sigma_\perp q_\perp + \sigma_3 M}{q_\perp^2 + M^2} \theta(q_-)\theta(p_- - q_-). \quad (128)$$

This is proportional to the expressions appearing in Eqs. (61), (86) and (100) in the main text since $p_- > 0$ and

$$(\sigma_\perp q_\perp + \sigma_3 M)^2 = q_\perp^2 + M^2. \quad (129)$$

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